Pocket Recap of Vector Calculus – Dr. Christopher J. Owen (UCL/MSSL)

(Not directly examinable, but maybe critical to understanding)

Introduction: The physics of plasmas involves the interaction of charged particles with magnetic and electric fields, which are vector quantities (i.e. they have both magnitude and direction dependencies). Moreover, fundamental properties such as the particles velocity are also vector quantities. Consequently, understanding the mathematics of plasmas involves many elements of what is known as 'Vector Calculus' – that is to say how gradients, derivatives and integrals can be applied to vector quantities themselves, or scalar quantities (scalar fields) that have a dependence on direction and whose gradients generally result in vector quantities (e.g. the gradient of a slope is not the same across the slope as it is down it). Many physics students will have (or should have) been exposed to vector calculus earlier in their college careers. However, for those that have either by-passed such course, or who have simply become rusty on the principles, we offer this basic summary of the main elements. Further reading on this subject can generally be found in good books with titles of the form 'Mathematics for Scientists/ Physicists /Engineers'.

In this sheet, unit vectors in the x, y, z direction are denoted **i**, **j**, **k**, respectively; the 'dot-product' or 'scalar product' of two vectors **A** and **B** is denoted **A.B**, while the 'cross-product' or 'vector product is denoted $\mathbf{A}\times\mathbf{B}$.

A. Derivatives

i) The *'del' operator*: $\underline{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is a vector operator;

ii) The *gradient* of a scalar field φ(x,y,z): $\underline{\nabla}\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z}$ (note the result is a vector);

iii) The *directional derivative* of φ in the direction of vector **a** is: $\frac{d\varphi}{ds} = \mathbf{a}.\underline{\nabla}\varphi$

iv) The *unit normal vector* to a surface defined by φ = const is: $\hat{\mathbf{n}} = \frac{\underline{\nabla}\varphi}{|\underline{\nabla}\varphi|}$ (this is the direction of the maximum rate of change of φ, which is $|\underline{\nabla}\varphi|$);

v) The *divergence* of a vector field **A**(x, y, z) is: $\text{div}\,\mathbf{A} = \underline{\nabla}.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ (note the result is a scalar);

vi) The *curl* of a vector field **A**(x, y, z) is: $\text{curl}\,\mathbf{A} = \underline{\nabla}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{d}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$ (note the result is a vector);

vii) A *conservative* or *irrotational* vector field **A** satisfies curl **A** = **0**;

viii) A *solenoidal* vector field A satisfies div **A** = 0;

ix) The *Laplacian* operator ('del-squared') is: $\nabla^2 = \underline{\nabla}.\underline{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$;

B. Integrals

i) *Position vector* along a curve:

$$\mathbf{r}(u) = \mathbf{i}x(u) + \mathbf{j}y(u) + \mathbf{k}z(u) \quad ; \quad d\mathbf{r} = \mathbf{i}\frac{\partial x}{\partial u}du + \mathbf{j}\frac{\partial y}{\partial u}du + \mathbf{k}\frac{\partial z}{\partial u}du$$

ii) *Line integrals of a scalar field* V(x,y,z) along path C between u1 and u2:

$$\int_C V d\mathbf{r} = \int_{u_1}^{u_2} V\left(\mathbf{i}\frac{\partial x}{\partial u} + \mathbf{j}\frac{\partial y}{\partial u} + \mathbf{k}\frac{\partial z}{\partial u}\right)du \quad \text{(the result is a vector);}$$

iii) *Line integrals of a vector field* **F**(x,y,z) along path C between u1 and u2:

$$\int_C \mathbf{F}.d\mathbf{r} = \int_{u_1}^{u_2} \left(F_x\frac{\partial x}{\partial u} + F_y\frac{\partial y}{\partial u} + F_z\frac{\partial z}{\partial u}\right)du \quad \text{(the result is a scalar);}$$

iv) Both above integrals depend on the path C chosen (unless curl F = 0 in (iii)). F and $\frac{d\mathbf{r}}{du}$ must be piecewise smooth functions;

C. Surface Integrals

i) Note *vector surface area element* is $d\mathbf{S} = \hat{\mathbf{n}}dS$

ii) *Surface integral of scalar field* V(x,y,z) is $\int_S V d\mathbf{S}$, the result is a vector;

iii) *Surface integral of vector field* **F**(x,y,z) is $\int_S \mathbf{F}.d\mathbf{S} = \iint_{S_x} F_x dydz + \iint_{S_y} F_y dxdz + \iint_{S_z} F_z dxdy$,

where S_x, S_y, S_z are projections of the surface S onto the x = 0, y = 0, z = 0 planes respectively. Note the projections must be unique, or the surface broken into sections which have a unique set of projections.

iv) *Stokes Theorem* states that the surface integral of the curl of a vector field over a surface S is equal to the line integral of the vector field around line C, the boundary of the surface S. Mathematically: $\iint_S (\underline{\nabla} \times \mathbf{F}).d\mathbf{S} = \oint_C \mathbf{F}.d\mathbf{r}$;

v) The Divergence Theorem says the volume integral of the divergence of a vector field over a volume V is equal to the surface integral of the vector field over the surface S bounding the volume V. Mathematically: $\iiint_V (\nabla.\mathbf{F})\, dV = \iint_S \mathbf{F}.d\mathbf{S}$;

D. Some Standard Vector Identities

Here φ and ψ are scalar fields, A and B are vector fields, C is a constant.

i) $\underline{\nabla}(\varphi + \psi) = \underline{\nabla}\varphi + \underline{\nabla}\psi$

ii) $\underline{\nabla}(C\varphi) = C\underline{\nabla}\varphi$

iii) $\underline{\nabla}.(\mathbf{A} + \mathbf{B}) = \underline{\nabla}.\mathbf{A} + \underline{\nabla}.\mathbf{B}$

iv) $\underline{\nabla}.(C\mathbf{A}) = C\underline{\nabla}.\mathbf{A}$

v) $\underline{\nabla} \times (\mathbf{A} + \mathbf{B}) = \underline{\nabla} \times \mathbf{A} + \underline{\nabla} \times \mathbf{B}$

vi) $\underline{\nabla} \times (C\mathbf{A}) = C\underline{\nabla} \times \mathbf{A}$

vii) $\underline{\nabla} \times \mathbf{r} = \mathbf{0}$, where r is the position vector $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

Analogues of the *Leibnitz product rule* for differentiation:

viii) $\underline{\nabla}(\varphi\psi) = \psi\underline{\nabla}\varphi + \varphi\underline{\nabla}\psi$

ix) $\underline{\nabla}.(\varphi\mathbf{A}) = \varphi\underline{\nabla}.\mathbf{A} + \mathbf{A}.(\underline{\nabla}\varphi)$

x) $\underline{\nabla} \times (\varphi\mathbf{A}) = \varphi\underline{\nabla} \times \mathbf{A} + (\underline{\nabla}\varphi) \times \mathbf{A}$

xi) $\underline{\nabla}.(\mathbf{A} \times \mathbf{B}) = \mathbf{B}.(\underline{\nabla} \times \mathbf{A}) - \mathbf{A}.(\underline{\nabla} \times \mathbf{B})$

xii) $\underline{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\underline{\nabla}.\mathbf{B}) - \mathbf{B}(\underline{\nabla}.\mathbf{A}) + (\mathbf{B}.\underline{\nabla})\mathbf{A} - (\mathbf{A}.\underline{\nabla})\mathbf{B}$

xiii) $\underline{\nabla}(\mathbf{A}.\mathbf{B}) = \mathbf{A} \times (\underline{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\underline{\nabla} \times \mathbf{A}) + (\mathbf{B}.\underline{\nabla})\mathbf{A} + (\mathbf{A}.\underline{\nabla})\mathbf{B}$

Note that $(\mathbf{B}.\underline{\nabla})$ and $(\mathbf{A}.\underline{\nabla})$ are *operators* which act on a vector. They have no meaning on their own.