Handout: Section 5.2 - Magnetohydrodynamics (MHD) – Equations (Examinable)

1. Convective Derivative

We need to describe how things change within a moving element of a fluid. The convective derivative lets us describe the way things change in a flowing fluid, while using a fixed reference frame.

Consider watching a small patch of sky. Clouds may develop in the patch as the water condenses out of the air, and will stay there if there is no wind. Clouds may be carried through the patch if there is a wind. Or both could happen; clouds may be carried through it from outside, while being added to due to cloud formation in the patch.

We describe change in a parameter A which goes on in a fixed region of space (whether or not the fluid is moving) using $\frac{\partial A}{\partial t}$.

We describe change in A due to the fluid element entering or leaving a region where A is different while moving along the x direction using $\left(\frac{\partial A}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right)$ where $\frac{\partial x}{\partial t}$ is the x component of the fluid flow velocity $\mathbf{u}$. This can be generalised to three dimensions as $(\mathbf{u}.\nabla)A$.

So the total rate of change of a scalar A in a fluid element in the flowing fluid is

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + (\mathbf{u}.\nabla)A$$

and the rate of change of a vector $\mathbf{A}$ in a fluid element in the flowing fluid is

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{A}$$

The $\frac{dA}{dt}$ term is the convective derivative, also sometimes written $\frac{DA}{Dt}$.

2. Pressure Tensor

The simplest form of pressure is isotropic pressure – the same in all directions. However, in a magnetised plasma, pressure may be different in the plane perpendicular to the field compared to the direction along the field (due to constraints on particle motion across the magnetic field). In principle, one could have shear stresses too, which give viscosity and oppose shear by transferring momentum in directions other than the direction of the momentum.

The pressure tensor represents pressure as a matrix. The leading diagonal terms contain the familiar pressure due to momentum flux in the direction of the momentum. The off-diagonal terms contain the less familiar shear stresses.

If we define one direction along the field, and the other two in the perpendicular plane, we can write the general pressure tensor as **P** = nm**vv** which, if there are no off-diagonal terms, is

$$\mathbf{P} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix}$$

where $p_\perp$ equals $p_\parallel$ in an isotropic plasma. It is usually more realistic to consider perpendicular and parallel components, since the fluid behaviour is only properly enforced in the perpendicular plane. Usually we can ignore off-diagonal terms.

For an anisotropic pressure, we define two temperatures, using the ideal gas law

$$p_\perp = nk_B T_\perp$$
$$p_\parallel = nk_B T_\parallel$$

3. Mass Conservation/Continuity Equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{v}) = 0$$

This tells us that the particle number density of the fluid in a region can only change if some of the fluid enters or leaves the region. For non-relativistic speeds (so that the particle masses are fixed) the equation also says that mass is conserved.

(Remember that Gauss' Theorem says that $\int \nabla.(n\mathbf{v})\, dV = \int n\mathbf{v}.\mathbf{dS}$ where the LHS is the integral of all elements dV in a volume V, enclosed by a surface S, and the RHS is the integral over all the enclosing surface elements **dS**.)

4. Charge Conservation/ Field Aligned Currents

The continuity equation also implies charge conservation;

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot (\rho_q\,\mathbf{v}) = 0$$

where ρ_q = qn, and one can express as:

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

If the time variation of the plasma conditions within a volume is small enough to ignore, so $\frac{\partial \rho_q}{\partial t} = 0$, then $\nabla . \mathbf{j} = 0$ so that $\nabla . \mathbf{j}_\perp = -\nabla . \mathbf{j}_\parallel$. In other words, currents must always close, so if there is a divergence of current in a plane perpendicular to the field, there must be field-aligned currents.

5. Equation of Motion

This equation tells us that momentum is conserved. It has the form ma = F, such that the terms on the RHS represent forces:

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\nabla . \underline{\underline{\mathbf{P}}} + \rho_q \mathbf{E} + \mathbf{j}\times\mathbf{B}$$

The LHS is convective derivative of momentum. Note that ρ = nm, the mass density, while the term ρ_q represents the charge density. However as a plasma is quasi-neutral we can generally ignore the $\rho_q\mathbf{E}$ term. The term **P** is the pressure tensor and **j** is the electric current density. Note that if we are dealing with a situation in which other forces are acting (e.g. gravity, see solar wind section later), then we must also include these forces on the RHS.

6. Generalized Ohm's Law

This equation tells us what can alter **j**.

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\ \mathbf{j} + \left(\frac{1}{ne}\right)(\mathbf{j}\times\mathbf{B}) - \left(\frac{1}{ne}\right)\nabla . \mathbf{P_e} + \left(\frac{m_e}{ne^2}\right)\frac{\partial \mathbf{j}}{\partial t}$$

The RHS terms are:

$\eta\mathbf{j}$	resistive term (like the ordinary Ohms law)
$\left(\frac{1}{ne}\right)(\mathbf{j}\times\mathbf{B})$	"Hall term" due to Lorentz force
$\left(\frac{1}{ne}\right)\nabla . \mathbf{P_e}$	term due to a possible anisotropic electron pressure
$\left(\frac{m_e}{ne^2}\right)\frac{\partial \mathbf{j}}{\partial t}$	term due to contribution of electron inertia to the current

In the special case of "ideal MHD", the resistivity vanishes, $\eta = 0$, so we lose the first term on the RHS. Note that in this case the fluid is a perfect conductor of electrical current. The restrictions on long length and timescales relating to applicability of MHD means that the 3rd and 4th terms are often small enough to be neglected. Making the additional assumption that currents perpendicular to the field are weak when ideal MHD is valid, we arrive at the conclusion:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

This corresponds to "frozen-in flow" (see lecture notes). Despite all these restrictions, the above equation is often a good approximation to reality for a space plasma.

7. Equation of State

In order to have a complete set of equations, we need to relate the pressure temperature and density. The simplest assumption is to use the ideal gas relation $p = n\, k_B T$. The assumption of constant temperature is often valid for very slowly changing plasma conditions, in which case $P \propto n$.

More rapid changes may justify the adiabatic assumption, in which no heat is exchanged between fluid elements. Then we may have $P \propto (mn)^{\gamma} = \rho^{\gamma}$ where γ can take various values according to the situation, but is often set at 5/3. The index γ is known as the adiabatic index, which is the ratio of specific heats. The MHD adiabatic equation of state is written:

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\left(P\rho^{-\gamma}\right) = 0$$

More rigorous (and thus more complicated) versions have sources and sinks of energy (heat flux, radiative terms, ohmic heating, depending on application) on the RHS.

8. Maxwells Equations

Note that the equations above may include a number of electromagnetic terms (**j, B, E**, etc.). Hence applying MHD to a given situation often requires the use of Maxwells equations to achieve full closure of the set of equations. Recall:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla . \mathbf{E} = \frac{\rho_q}{\varepsilon_o}$$

$$\nabla \times \mathbf{B} = \mu_o\, \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla . \mathbf{B} = 0$$

where **E** is the electric field vector, **B** the magnetic field vector, $c^2=(\varepsilon_o\mu_o)^{-1}$, ρ_q is the charge density, and ***j*** is the current density.