Space Plasma and Magnetospheric Physics

Section 6.1

A Simple (MHD) Model of the Solar Wind (Parker Model)

Prelims

- We will try to understand the gross/average properties of the solar wind
- We will look for steady-state solutions of our MHD equations
- We will make a number of simplifying assumptions

MHD Equations

Mass Conservation/ Continuity Equation:

Assumption 1: Steady State, $\frac{\partial}{\partial t} = 0$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \quad \equiv \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Equation of Motion:

Assumption 2: Neglect electromagnetic forces c.f. required gravity forces

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} = -\nabla \mathbf{P} + \rho_q \mathbf{E} + \mathbf{j} \times \mathbf{B} + \boxed{\rho \mathbf{F}_g}$$

Equation of State:

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\left(P\rho^{-\gamma}\right) = 0$$

Assumption 3: Isothermal expansion of an ideal gas: $\gamma = 1$

Assumption 4: Radially symmetric solution – all parameters functions of r only

Substitutions 1

- Radial symmetry, use spherical polar coordinates with: $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0$

- Hence: $\nabla P = \frac{dP}{dr}\hat{\mathbf{r}}$

$$\nabla \cdot (\rho \mathbf{u}) = \frac{1}{r^2}\frac{d}{dr}\left(\rho u r^2\right)$$

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \rho u \frac{du}{dr}\hat{\mathbf{r}}$$

- Solution of the form: $\mathbf{u} = u(r)\,\hat{\mathbf{r}}$

Substitutions 2

- Gravity dominates forces, use:

$$\mathbf{F}_g = \frac{-GM_S}{r^2}\hat{\mathbf{r}}$$

- Isothermal expansion of ideal gas, Equation of State becomes:

$$\frac{P}{\rho} = Const. = \frac{2kT}{m}$$

$$\left(T = \frac{(T_i + T_e)}{2}; \quad m = m_i + m_e; \quad \rho = nm \right)$$

3 Equations, 3 Unknowns

- Mass Equation ▶ $$\frac{d}{dr}\left(\rho u r^2\right) = 0 \qquad (1)$$
- Momentum Eqn ▶ $$\rho u \frac{du}{dr} = -\frac{dP}{dr} - \rho \frac{GM_S}{r^2} \qquad (2)$$
- Eqn of State ▶ $$P = \frac{2kT}{m}\rho \qquad (3)$$

Now look for solutions for *u(r)*

a) Why not a Static Atmosphere?

- Clearly $u(r) = 0$ is a solution to (1).
  - (2)→

$$-\frac{dP}{dr} - \rho \frac{GM_s}{r^2} = 0$$

  - Pressure gradient is balanced by gravity!
- Sub for ρ from (3):

$$\frac{1}{P}\frac{dP}{dr} = -\frac{GM_S m}{2kT}\frac{1}{r^2}$$

$$\Rightarrow \quad P(r) = P_o \exp\left[\frac{GM_S m}{2kT}\left\{\frac{1}{r} - \frac{1}{R}\right\}\right]$$

  - where $P = P_o$ at $r = R$, the base of the corona.

a) Why not a Static Atmosphere?

$$P(r) = P_o \exp\left[\frac{GM_S m}{2kT}\left\{\frac{1}{r} - \frac{1}{R}\right\}\right]$$

- This solution is OK for a shallow atmosphere (where $r - R \ll R$ everywhere)
  - (e.g. thin shell of Earths atmosphere).
- But solar atmosphere is *not* shallow, so for large *r*:

$$P(r) = P_o \exp\left[-\frac{GM_S m}{2kT}\frac{1}{R}\right]$$

- RHS is not small enough to account for low pressures in interplanetary space!

b) So we need a *u(r) ≠ 0* solution:

- (1) gives $\rho u r^2 = C \quad or \quad \rho = \dfrac{C}{u r^2} \quad (4)$
- Differentiate (3) w.r.t. *r* :

$$\frac{dP}{dr} = \frac{2kT}{m}\frac{d\rho}{dr} = \frac{2kT}{m} C \frac{d}{dr}\left(\frac{1}{ur^2}\right) \quad from\ (4)$$

$$= \frac{2kT}{m} C \left(-\frac{1}{u^2 r^2}\frac{du}{dr} - \frac{1}{u}\frac{2}{r^3} \right)$$

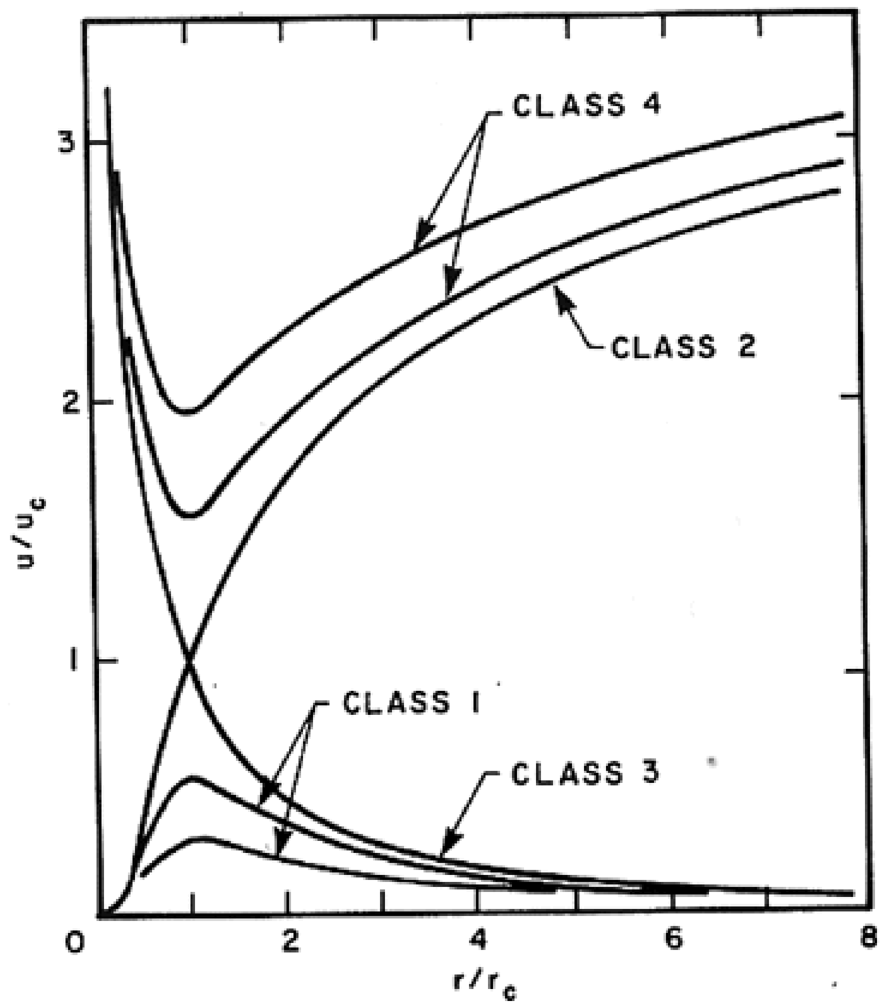
- So substitute for *ρ* and *dP/dr* into (2), rearrange and find (exercise):

$$\left[u^2 - \frac{2kT}{m} \right] \frac{1}{u}\frac{du}{dr} = \frac{4kT}{mr} - \frac{GM_S}{r^2}$$

O.D.E. for *u*

Solutions

- Finding mathematical solutions is a straightforward exercise for you.
- Graphical representation of 4 possible solutions:



Solutions (2)

$$\left[u^2 - \frac{2kT}{m}\right]\frac{1}{u}\frac{du}{dr} = \frac{4kT}{mr} - \frac{GM_S}{r^2}$$

- Lets look at physical meaning of the maths:
  - In corona (small *r*), RHS < 0 (gravity dominates)
  - At large *r*, RHS > 0 (gravity term falls off faster)
- Hence RHS = 0 at:

$$r = r_C = \frac{GM_S m}{4kT}$$

Critical Radius

Solutions (3)

$$\left[u^2 - \frac{2kT}{m}\right]\frac{1}{u}\frac{du}{dr} = \frac{4kT}{mr} - \frac{GM_S}{r^2}$$

- Now look at LHS:
  - In corona (small *r*) *u* is small, so

$$\left[u^2 - \frac{2kT}{m}\right] < 0$$

$$\Rightarrow \frac{du}{dr} > 0 \quad for \quad r < r_C$$

  - Hence in this region *u* increases as *r* increases (rules out class 3 and 4 solutions in Figure above).

Solutions (4)

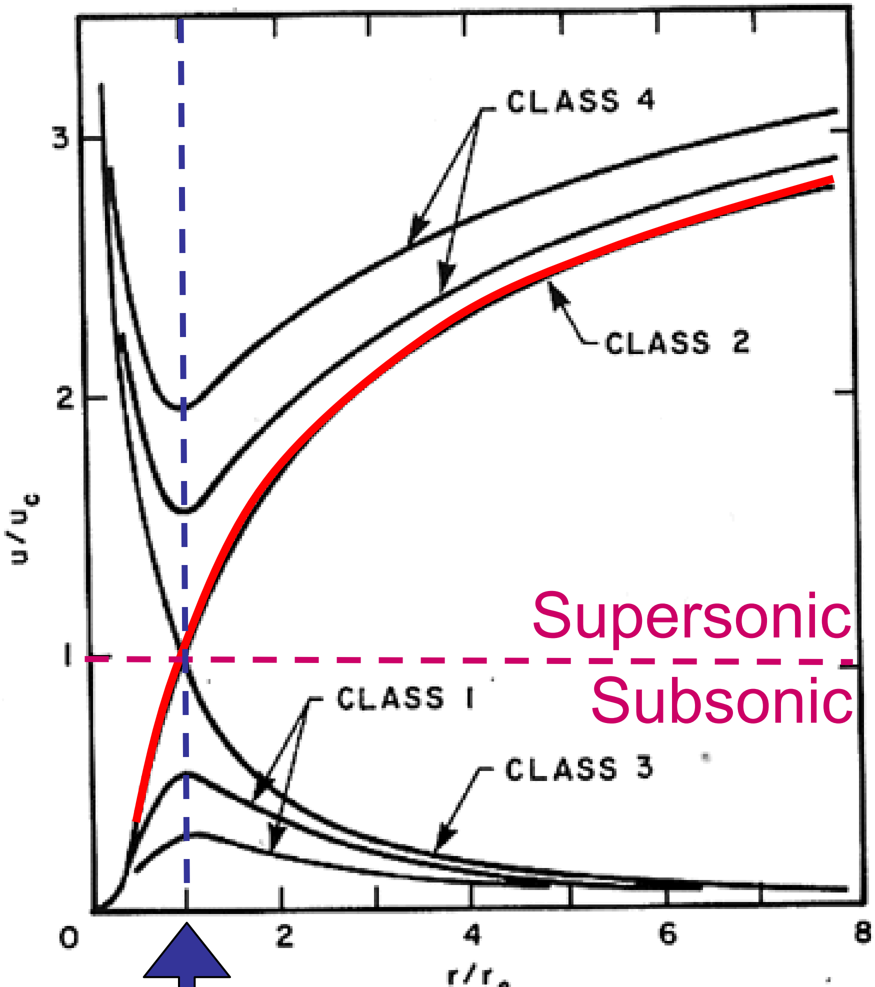
$$\left[u^2 - \frac{2kT}{m}\right]\frac{1}{u}\frac{du}{dr} = \frac{4kT}{mr} - \frac{GM_S}{r^2}$$

- At $r = r_C$, RHS = 0, so either:
  - $du/dr = 0$, which implies $du/dr < 0$ for $r > r_C$
    - (Similar to static atmosphere solution, Class 1 above)
  - Or $\left[u^2 - \frac{2kT}{m}\right] = 0$

    $$\Rightarrow \frac{du}{dr} > 0 \quad still \;\; for \quad r > r_C$$

    - u continues to increase with increasing r (Class 2)
    - N.B. $\rho(r) = C/ur^2 \rightarrow 0$ as both u and r increase, reducing pressure to low values consistent with observations

Solutions (5)



$r = r_C$ (a few R_S)

- This solution has

$$\left[u^2 - \frac{2kT}{m}\right] = 0$$

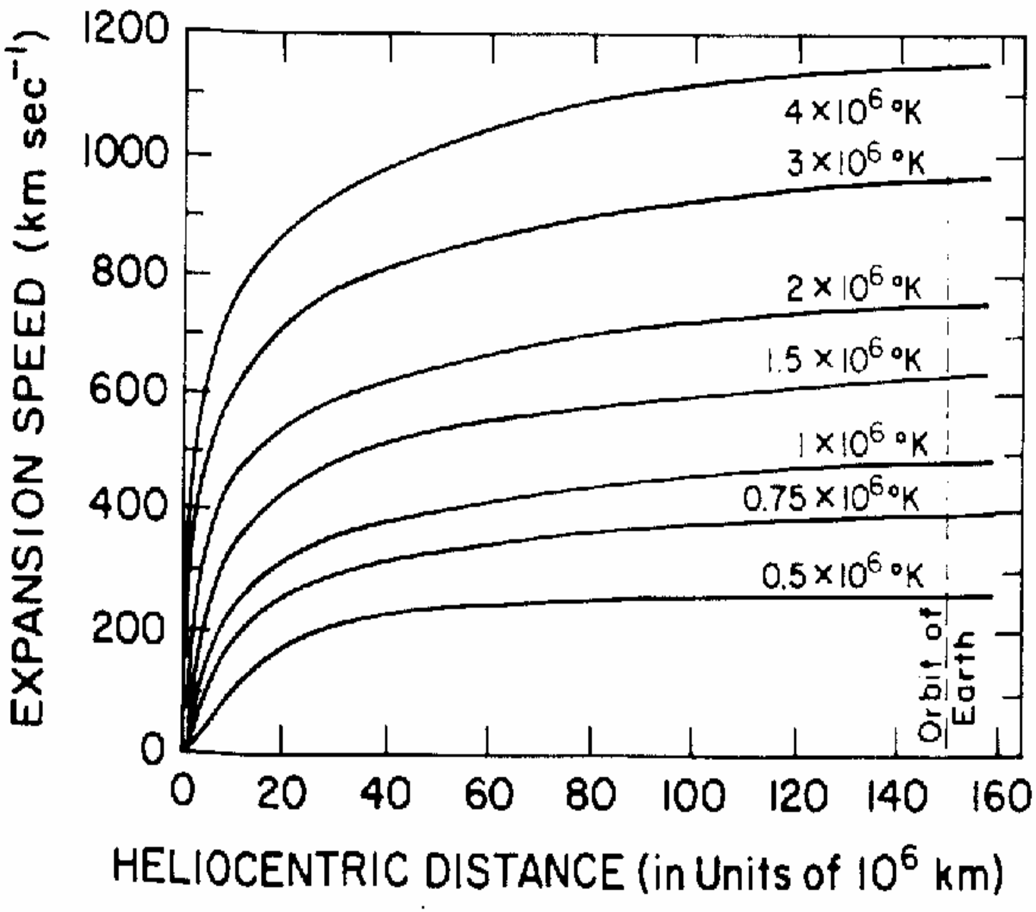
$$at \quad r < r_C. \quad Or$$

$$u = \sqrt{\frac{2kT}{m}} = C_S$$

$$for\ \gamma = 1\ gas$$

- So flow becomes supersonic at $r = r_C$ (and switches from gravity bound)

Solutions (6)



- Solar wind is a continuous, supersonic outflow of plasma from the solar corona
- Flow reaches a ~constant velocity at large distances

Summary, Section 6.1

- The average properties of the solar wind can be obtained from a simple (M)HD model, giving continuous supersonic outflow.
- This model *does not* address several important solar wind questions, e.g.
  - What causes flow acceleration?
  - Where does required energy come from?
  - What is the physics behind the very special mathematical solution found here? Why should it apply in nature?