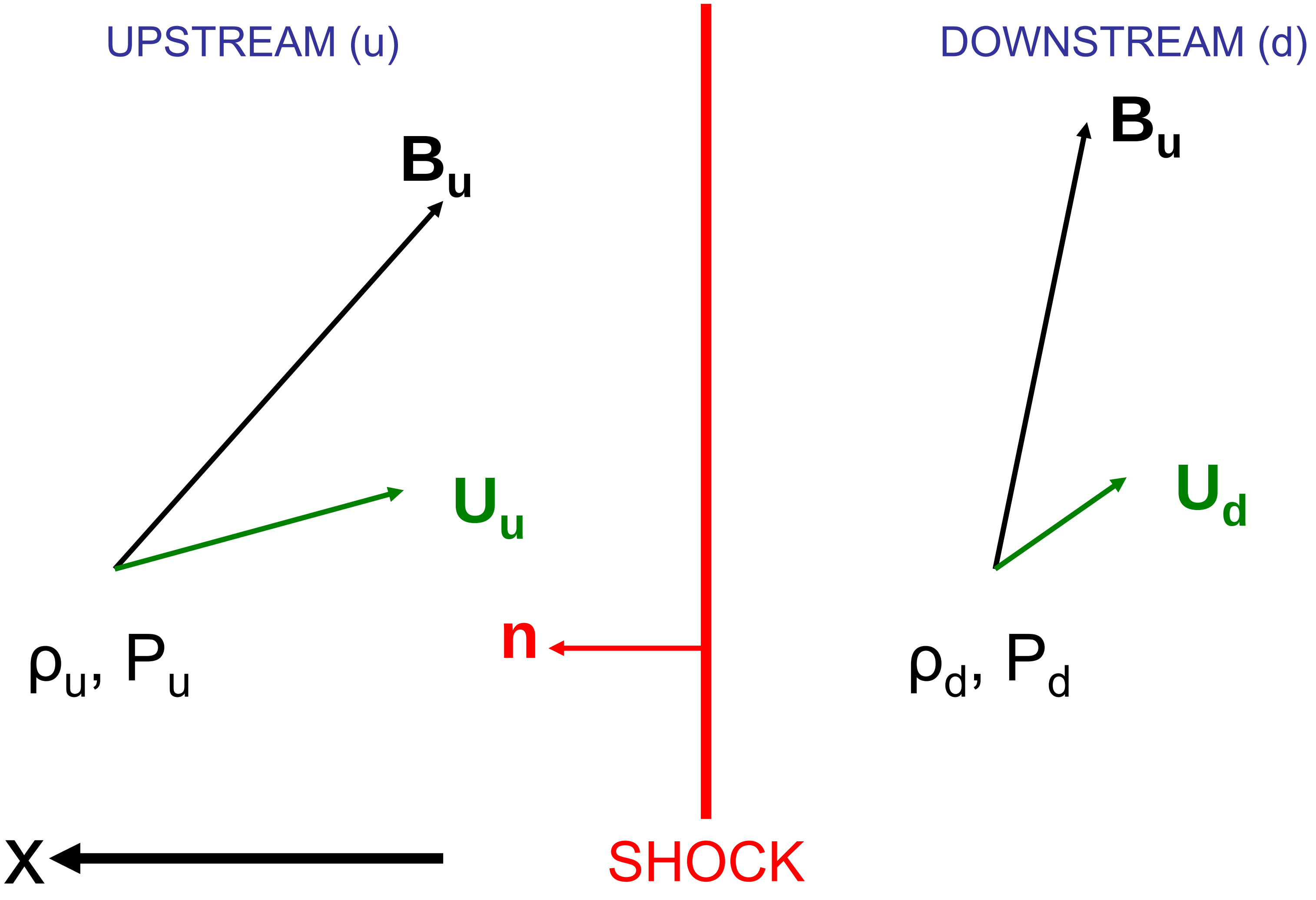
Rankine-Hugoniot Shock Jump Conditions

Shock Geometry



Assumptions

- Plasma is everywhere isotropic ($p_{||} = p_{\perp}$);
- Shock is 1-d (curvature $R_c >> d$, thickness);
- Shock lies in the Y-Z plane (i.e. the shock normal n is parallel to the x-direction);
- Thus $\partial/\partial y = \partial/\partial z = 0$;
- The shock is a sharp discontinuity ($d \rightarrow 0$);
- Shock is in steady state ($\partial/\partial t = 0$);
- We work in the rest frame of the shock;

- Notation: $[X] = X_u - X_d$ for any quantity X.

Mass Conservation

- Mass flowing into the shock must exit from the other side!
- Normal mass flux is ρu_x
- The MHD mass conservation or continuity equation for e.g. number density *n*:

$$\left.\begin{array}{l} \cancel{\dfrac{\partial n}{\partial t}} + \nabla \cdot (n\,\mathbf{v}) = 0 \\[2ex] \dfrac{\partial}{\partial x}(\rho u_x) = 0 \end{array}\right\} \quad [\rho u_x] = 0$$

$\boxed{\text{RH1}}$

This says if the plasma slows it also compresses

Momentum Conservation

$$\underbrace{\rho\left(\cancel{\frac{\partial}{\partial t}}+\mathbf{v}\cdot\nabla\right)\mathbf{v}=-\nabla.\underline{\underline{\mathbf{P}}}+\cancel{\rho_q\mathbf{E}}}-\underbrace{\nabla\left(\frac{B^2}{2\mu_0}\right)+\frac{1}{\mu_0}(\mathbf{B}.\nabla)\mathbf{B}}_{}$$

Using our magnetic pressure and tension split for the magnetic force **j** x **B**

- Momentum is a vector, so generally has normal (n) and transverse (t) components;
- So split other vector quantities B and u in the same way, i.e.

$$\mathbf{u}=u_x\mathbf{n}+\mathbf{u}_\mathbf{t}$$

$$\mathbf{B}=B_x\mathbf{n}+\mathbf{B}_\mathbf{t}$$

Momentum Conservation

- Consider first normal momentum:

$$\underbrace{\rho\left(u_x \frac{\partial}{\partial x}\right)u_x}_{\text{Change of normal momentum flux}} + \underbrace{\frac{\partial P}{\partial x}}_{\text{Plasma pressure gradient}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{B^2}{2\mu_0}\right)}_{\text{Magnetic pressure gradient}} + \underbrace{\frac{1}{\mu_0}\left(\cancel{B_x \frac{\partial}{\partial x}}\right)B_x}_{\text{We will show this term is zero below in RH5 below, since div }\mathbf{B}\text{ = 0}} = 0$$

- Using RH1:

$$\left[\rho u_x^2 + P + \frac{B^2}{2\mu_0}\right] = 0 \qquad \boxed{\text{RH2}}$$

- Loss of normal momentum gives rise to increase in pressure(s)

Momentum Conservation

- Now consider transverse momentum:

$$\underbrace{\rho\left(u_x \frac{\partial}{\partial x}\right)\mathbf{u_t}}_{\text{Change of transverse momentum flux}} + \underbrace{\frac{1}{\mu_0}\left(B_x \frac{\partial}{\partial x}\right)\mathbf{B_t}}_{\text{Change in } \mathbf{B}_t \text{ across shock means a magnetic tension in a bent field line}} = 0$$

- Use RH1 and RH5 again:

$$\left[\rho u_x \mathbf{u_t} + \frac{B_x \mathbf{B_t}}{\mu_0}\right] = \mathbf{0}$$

RH3

- Transverse momentum gives rise to increase in magnetic tension (field line bending)

Conservation of Energy

- At shocks energy can be converted between different forms:
  - Most relevant to space plasmas:
    - Plasma kinetic energy: $\frac{1}{2}mV^2$
    - plasma internal energy: $P\rho^{-\gamma} = const$
    - magnetic energy flux: $\mathbf{S} = \frac{(\mathbf{E}\times\mathbf{B})}{\mu_0} = \frac{1}{\mu_0}(-\mathbf{u}\times\mathbf{B}\times\mathbf{B})$

$$\left[\rho u_x \left(\underbrace{\frac{u^2}{2}}_{\text{KE flux}} + \underbrace{\frac{\gamma}{\gamma-1}\frac{P}{\rho}}_{\text{Enthalpy}}\right) + \underbrace{u_x\frac{B^2}{\mu_0} - (u.B)\frac{B_n}{\mu_0}}_{\text{Poynting Flux}}\right] = 0$$

RH4

Maxwells Equations

- Fields only conditions:

$$\left.\begin{array}{l} \nabla \,.\, \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\cancel{\dfrac{\partial \mathbf{B}}{\partial t}} = 0 \end{array}\right\} \begin{array}{l} [B_x] = 0 \quad \boxed{\text{RH5}} \\ [\mathbf{E_t}] = [u_x \mathbf{B_t} - B_x \mathbf{u_t}] = \mathbf{0} \quad \boxed{\text{RH6}} \end{array}$$

  - RH5 – the normal component of B is conserved;
  - RH6 – the tangential component of E is conserved;
- 6 Equations, 6 unknowns (ρ, P, u_n, $\mathbf{u_t}$, B_n, $\mathbf{B_t}$), thus given upstream values we can calculate all downstream values

Some Notes

- The above is for 1-fluid ideal MHD
  - Anisotropic pressure ($P_{||} \neq P_{\perp}$)
  - Different species (electrons, ions)

  need further assumptions to close set of eqns.
- These equations hold for any type of discontinuity in the plasma (current sheets, density gradients, etc. – see table 5.2 in K&R). *For shocks we need flow through the discontinuity* $u_n \neq 0$.

Fast Mode Shocks

- Form from a steepening of the *fast mode magnetosonic wave*, which has plasma pressure variations in phase with magnetic field strength variations
  - P↑ B↑ but B_n = const, hence $|B_t|$↑
- These are the most common type of shock occurring in space plasmas (bow shock, interplanetary shocks are fast shocks)

Slow Mode Shocks

- Form from a steepening of the *slow mode magnetosonic wave*, which has plasma pressure variations out of phase with magnetic field strength variations
  - P↑ B↓ but B_n = const, hence $|B_t|$↓
- These are rarer than fast shocks, but arise in some models of magnetic reconnection (a subject we shall discuss in later lectures)

Magnetic Co-planarity

For both fast and slow shocks, the Co-planarity theorem states that the vectors $\mathbf{B_u}$, $\mathbf{B_d}$ and $\mathbf{n}$ all lie in the same plane. Mathematically:

$$\mathbf{n}.\left(\mathbf{B_u} \times \mathbf{B_d}\right) = 0$$

(This is really useful in analysing data)